Nonequilibrium stochastic processes: Time dependence of entropy flux and entropy production

Bidhan Chandra Bag*
Department of Chemistry, Visva-Bharati, Santiniketan 731 235, India
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Based on the Fokker-Planck and the entropy balance equations we have studied the relaxation of a dissipative dynamical system driven by external Ornstein-Uhlenbeck noise processes in the absence and presence of nonequilibrium constraint in terms of the thermodynamically inspired quantities such as entropy flux and entropy production. The interplay of nonequilibrium constraint, dissipation, and noise reveals some interesting extremal nature in the time dependence of entropy flux and entropy production.

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I. INTRODUCTION

Understanding of the nature of nonequilibrium and equilibrium states of a dynamical system in presence of surroundings is always an intriguing issue of physics. Entropy is an important quantity in this regard in thermodynamics. While in the traditional classical thermodynamics, the specific nature of a stochastic process is irrelevant, this may play an important role for establishing the connection between the phase space of a dynamical system and the related thermodynamically inspired quantities such as entropy production, flux, and Onsagar coefficients, etc. Recently a number of authors [1–11] have explored the relationship in considerable detail.

The aim of the present paper is to enquire into this connection about the imprints of color [12], white, and cross-correlated noise processes [13,14] on time dependence of entropy, entropy production, and entropy flux using a connection between the information entropy and the probability distribution function of the phase-space variables for thermodynamically open systems. Based on a Fokker-Planck description of stochastic processes and the entropy balance equation we first consider here the relaxation of a dissipative dynamical system in presence of the noise processes to a steady state from a given nonequilibrium state in terms of thermodynamically inspired quantities. For additive white noise we compare our results in the equilibrium state with the standard results for the closed systems. We also enquire how the system relaxes if the system is thrown away from the aforesaid steady state by a nonequilibrium constraint to understand how the entropy flux and the entropy production pass through minima with time in the latter case and how the two relaxation processes for different noise properties differ.

The outline of the paper is as follows: In Sec. II we calculate the entropy flux and the entropy production for a simple dissipative dynamical system in the nonequilibrium state for different noise processes. The paper is concluded in the Sec. III.

*Email address: pcbcb@yahoo.com

II. THE FOKKER-PLANCK DESCRIPTION, TIME DEPENDENCE OF ENTROPY FLUX AND PRODUCTION OF NOISE-DRIVEN DYNAMICAL SYSTEMS

A. Relaxation of the noise-driven dynamical system to the steady state

1. Ornstein-Uhlenbeck noise process

We consider the dynamics of a dissipative dynamical system driven by the external Ornstein-Uhlenbeck noise process in the phase space. The relevant Langevin equation of motion can be written as

$$\dot{X} = -\gamma X + \eta, \tag{1}$$

where γ is the damping constant. The term η in Eq. (1) is the external Ornstein-Uhlenbeck noise whose two time correlation is given by

$$\langle \eta(t)\,\eta(t')\rangle = \frac{D}{\tau}\exp\left(-\frac{|t-t'|}{\tau}\right). \tag{2}$$

D is the noise strength and τ corresponds to the correlation time of color noise process. The time evolution of η can be conveniently expressed in terms of the Gaussian white noise process $\zeta(t)$ as

$$\dot{\eta} = -\frac{\eta}{\tau} + \frac{\sqrt{D}}{\tau}\zeta, \tag{3}$$

$$\langle \zeta(t)\zeta(t')\rangle = 2\epsilon\delta(t-t'),$$

and

$$\langle \zeta \rangle = 0,$$

where the parameter ϵ is used to identify the noise strength.

Now treating η as a phase-space variable on the same footing as X we can write Fokker-Planck equation in the extended phase space [12] as

$$\frac{\partial \rho(X_1,X_2,t)}{\partial t} = \gamma\frac{\partial X_1\rho}{\partial X_1} - X_2\frac{\partial \rho}{\partial X_1} + \frac{1}{\tau}\frac{\partial X_2\rho}{\partial X_2} + \epsilon\frac{D}{\tau^2}\frac{\partial^2\rho}{\partial X_2^2}, \tag{4}$$

where X_1, X_2 refer to X and η in Eq. (1) and $\rho(X_1, X_2, t)$ is the extended phase-space probability distribution function.

Now making use of the transformation

$$U = aX_1 + X_2, \tag{5}$$

the Fokker-Planck Eq. (4) can be written as

$$\frac{\partial \rho(U,t)}{\partial t} = -\frac{\partial F \rho}{\partial U} + \epsilon D_s \frac{\partial^2 \rho}{\partial U^2}, \tag{6}$$

where

$$F = -\lambda U, \tag{7}$$

$$\lambda U = \gamma a X_1 - a X_2 + \frac{X_2}{\tau}, \tag{8}$$

and

$$D_s = \frac{D}{\tau^2}. \tag{9}$$

Here a and λ are constants to be determined. Using Eq. (5) in Eq. (8) and comparing the coefficients of X_1 and X_2 we find

$$\lambda = \gamma \quad \text{and} \quad a = \frac{1 - \gamma\tau}{\tau}. \tag{10}$$

We are now in a position to define entropy flux and entropy production using Eq. (6). In the microscopic picture the Shannon form of the entropy is connected to the continuous probability distribution ρ as

$$S = -\int \rho(U,t) \ln \rho(U,t)\, dU. \tag{11}$$

The time evolution equation for entropy then can be written as

$$\frac{dS}{dt} = -\int dU \left[-\frac{\partial F \rho}{\partial U} + \epsilon D_s \frac{\partial^2 \rho}{\partial U^2} \right] \ln \rho. \tag{12}$$

Putting the usual boundary conditions into the result of partial integration of the right-hand side of the above Eq. (12), one obtains the following form of information entropy balance:

$$\frac{dS}{dt} = \int \rho \frac{\partial F}{\partial U} dU + \epsilon D_s \int \frac{1}{\rho} \left(\frac{\partial \rho}{\partial U} \right)^2 dU. \tag{13}$$

Equation (13) implies that the first term has no definite sign while the second term is positive definitely since D_s is always positive. Then one can identify the first and the second terms as entropy flux $\dot{S}_F$ and entropy production $\dot{S}_P$, respectively:

$$\dot{S}_F = \int \rho \frac{\partial F}{\partial U} dU, \tag{14}$$

$$\dot{S}_P = \epsilon D_s \int \frac{1}{\rho} \left(\frac{\partial \rho}{\partial U} \right)^2 dU. \tag{15}$$

To find the explicit time dependence of these quantities we then search for the Green's function or conditional probability solution for the system at U at time t for the given initial condition,

$$\rho(U, t=0) = \frac{\epsilon_1}{\pi} \exp[-\epsilon_1 (U - U')^2]. \tag{16}$$

We now look for a solution of Eq. (6) of the form

$$\rho(U, t | U', 0) = \exp[G(t)], \tag{17}$$

where

$$G(t) = -\frac{1}{\sigma(t)} [U - \beta(t)]^2 + \ln \nu(t). \tag{18}$$

We will see that by suitable choice of $\beta(t)$, $\sigma(t)$, $\nu(t)$ one can solve Eq. (6) subject to the initial condition,

$$\rho(U, 0 | U', 0) = \frac{\epsilon_1}{\pi} \exp[-\epsilon_1 (U - U')^2]. \tag{19}$$

Comparing Eq. (19) with Eq. (17) and $G(0)$ we have

$$\sigma(0) = \frac{1}{\epsilon_1}, \quad \beta(0) = U', \quad \nu(0) = \frac{\epsilon_1}{\pi}. \tag{20}$$

If we put Eq. (17) in Eq. (6) and equate the coefficients of equal powers of U we obtain after some algebra the following set of equations:

$$\dot{\sigma}(t) = -2\gamma \sigma(t) + 4 \epsilon D_s, \tag{21}$$

$$\dot{\beta}(t) = -\gamma \beta(t), \tag{22}$$

$$\frac{1}{\nu(t)} \dot{\nu}(t) = -\frac{1}{2\sigma(t)} \dot{\sigma}(t). \tag{23}$$

The relevant solutions of $\sigma(t)$ and $\beta(t)$ for the present problem which satisfy the initial conditions above are given by

$$\sigma(t) = \frac{2\epsilon D_s}{\gamma} [1 - \exp(-2\gamma t)] + \sigma(0) \exp(-2\gamma t) \tag{24}$$

and

$$\beta(t) = \beta(0) \exp(-\gamma t). \tag{25}$$

Now making use of Eqs. (17), (24), and (25) in Eqs. (14) and (15) we finally obtain the explicit time dependence of the entropy flux and the entropy production as

$$\dot{S}_F = -\gamma \tag{26}$$

and

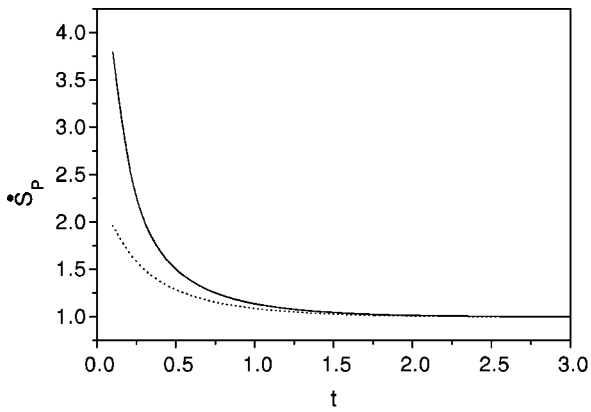

FIG. 1. Plot of entropy production $\dot{S}_P$ vs time using Eq. (27) for $\sigma(0)=0.1$, $D=0.5$, and $\gamma=1.0$. Solid and dotted curves are for $\tau=2$ and 1, respectively (units are arbitrary).

$$\dot{S}_P=\frac{2\epsilon D}{\tau^2\left[\frac{2\epsilon D}{\gamma\tau^2}+\left\{\sigma(0)-\frac{2\epsilon D}{\gamma\tau^2}\right\}\exp(-2\gamma t)\right]}, \qquad (27)$$

respectively, where we have used $D_s=D/\tau^2$. Thus entropy flux is negative and is independent of time, noise strength, and correlation time. But entropy production decreases monotonically almost exponentially with time for a given set of D, τ, and γ as shown in Fig. 1 and finally reaches to the limiting value γ at the long time satisfying ($\dot{S}_F=-\dot{S}_P$)

$$\frac{dS}{dt}=\dot{S}_F+\dot{S}_P=0. \qquad (28)$$

We now examine the connection between the thermodynamic entropy production and the phase-space collapse of the systems in nonequilibrium stationary states. In this state $dS/dt=0$ and we have from Eqs. (13), (14), and (15) (for details see Ref. [7])

$$\dot{S}_P=-\dot{S}_F=-\int\rho\frac{\partial F}{\partial U}dU=-\overline{\mathrm{div}F^{\infty}}=-\sigma'+O(\epsilon)>0 \qquad (29)$$

in the limit $\epsilon\ll 1$. Here σ' is the Lyapunov exponent of the one-dimensional deterministic system. Thus information entropy as defined by Eq. (15) is equal to the negative of Lyapunov exponent or equivalently to the rate of phase-space volume contraction plus a correction term vanishing as the noise strength goes to zero [15,16]. The results in Eq. (29) is very much interesting, since it would seem at first sight from Eq. (15) that $\dot{S}_P$ should tend to zero as $\epsilon\to 0$. The fact is that it nevertheless gives a finite contribution in this limit, which reflects the nonanalytic dependence of the probability density on ϵ [7].

2. Cross-correlated noise process

We now consider another case where a simple dissipative system is driven by both additive and multiplicative white Gaussian noises,

$$\dot{X}=-\gamma X-\zeta_1 X+\eta_1. \qquad (30)$$

The correlation between the noise processes are given by

$$\langle\zeta_1(t)\zeta_1(t')\rangle=2\epsilon D'\,\delta(t-t'),$$

$$\langle\eta_1(t)\,\eta_1(t')\rangle=2\epsilon\alpha\,\delta(t-t'),$$

$$\langle\zeta_1(t)\,\eta_1(t')\rangle=\langle\zeta_1(t')\,\eta_1(t)\rangle=2\lambda_1\epsilon\sqrt{D'\alpha}\,\delta(t-t'),$$

$$0\leqslant\lambda_1\leqslant 1, \qquad (31)$$

where λ_1 denotes the cross correlation of the two noise processes. The Fokker-Planck equation for the Langevin Eq. (30) can be written as (for details see Ref. [8])

$$\frac{\partial\rho}{\partial t}=-\frac{\partial F\rho}{\partial X}+\epsilon D_1\frac{\partial^2\rho}{\partial X^2}, \qquad (32)$$

where the drift term is

$$F=-\Gamma X+l \qquad (33)$$

and

$$D_1=[\alpha\gamma^2+(2-\nu)\epsilon D'\alpha\{(2-\nu)\epsilon D'+2\gamma-2\gamma\lambda_1^2-\lambda_1^2(2-\nu)\epsilon D'\}]/\Gamma^2 \qquad (34)$$

with

$$\Gamma=\gamma+2\epsilon D'-\nu, \quad l=(2-\nu)\lambda_1\epsilon\sqrt{D'\alpha}. \qquad (35)$$

In Eqs. (34) and (35) $\nu=1$ stands for the Stratonovich convention and $\nu=0$ for the Ito convention.

The Fokker-Planck equation (32) is very similar to Eq. (6). Following the earlier method the time dependence of entropy flux and entropy production for the cross-correlated noise-driven process is

$$\dot{S}_F=-\Gamma, \qquad (36)$$

$$\dot{S}_P=\frac{2D_1}{\sigma_1(t)}, \qquad (37)$$

where

$$\sigma_1(t)=\frac{2\epsilon D_1}{\Gamma}+\left(\sigma_1(0)-\frac{2\epsilon D_1}{\Gamma}\right)\exp(-2\Gamma t). \qquad (38)$$

Here $\sigma_1(0)$ has the same significance as in Eq. (24). Thus entropy flux for the cross-correlated noise process is time independent but its value not only depends on dissipation constant γ as in the previous case but also on the strength of multiplicative noise D'. The time dependence of entropy production is qualitatively same as in the Fig. 1 but the relaxation time is different since Γ contains both γ and D'. In the long time limit Eqs. (36) and (37) satisfy Eq. (28). Since Eqs. (6) and (32) are formally same, the connection between the thermodynamic entropy production and the phase-space collapse of systems in nonequilibrium stationary states for the correlated noise-driven system should be similar to Eq. (29). Using $D'=0$, $\lambda_1=0$, $\nu=0$, and $\alpha=\gamma KT$ in Eq. (37) (K and T are Boltzmann constant and temperature, respectively) one can obtain the time dependence of entropy flux and production for thermodynamically closed system [17] in the Markovian limit.

B. Relaxation of the small external force-driven steady state to the new steady state

1. The Ornstein-Uhlenbeck noise process

We shall now examine the time dependence of entropy flux and production during the relaxation of steady state to a new steady state for the system driven by an weak external force. To this end we consider the constant drift f_e in Eq. (1) due to external force so that the total drift in Eq. (6) now becomes

$$F=F_0(U)+hF_1, \tag{39}$$

where $F_0=-\lambda U$, $F_1=af_e$, and h is smallness parameter. When $h=0$, $\rho=\rho_s$, ρ_s is the steady state solution of the Eq. (6). The deviation of ρ from ρ_s in the presence of nonzero small h can be explicitly taken into account once we make use of the identity for the diffusion term in Eq. (6),

$$\frac{\partial^2\rho}{\partial U^2}=\frac{\partial}{\partial U}\left[\rho\frac{\partial \ln\rho_s}{\partial U}\right]+\frac{\partial}{\partial U}\left[\rho_s\frac{\partial}{\partial U}\frac{\rho}{\rho_s}\right]. \tag{40}$$

Now we are in a position to establish a connection between the entropy production of irreversible thermodynamics and the relevant quantities of the underlying dynamics in phase space for the present model following Ref. [7]. The explicit calculation using Eq. (40) shows that the information entropy balance Eq. (12) now yields

$$\begin{aligned}\frac{dS}{dt}=&-\int dU\ln\rho\left[-\frac{\partial(F\rho)}{\partial U}+\epsilon D_s\frac{\partial}{\partial U}\left(\frac{\rho\partial\ln\rho_s}{\partial U}\right)\right]\\&-\epsilon D_s\int dU\ln\rho_s\frac{\partial}{\partial U}\left(\rho_s\frac{\partial}{\partial U}\frac{\rho}{\rho_s}\right)\\&+\epsilon D_s\int dU\rho\left(\frac{\partial}{\partial U}\ln\frac{\rho}{\rho_s}\right)^2.\end{aligned} \tag{41}$$

It is noted that the first, the second, and the third integrals in Eq. (41) are of zeroth, first, and second order, respectively, with respect to the deviation from steady state. Doing partial integrations in Eq. (41) we obtain

$$\begin{aligned}\frac{dS}{dt}=&\overline{\mathrm{div}F^t}+\epsilon D_s\int dU\rho\left[-\left(\frac{\partial\ln\rho_s}{\partial U}\right)^2+2\frac{\partial\ln\rho}{\partial U}\frac{\partial\ln\rho_s}{\partial U}\right]\\&+\epsilon D_s\int dU\rho\left(\frac{\partial}{\partial U}\ln\frac{\rho}{\rho_s}\right)^2.\end{aligned} \tag{42}$$

Such a new decomposition of the rate of change of information entropy now exhibits a part $\Delta\dot{S}_P$,

$$\Delta\dot{S}_P=\epsilon D_s\int dU\rho\left(\frac{\partial}{\partial U}\ln\frac{\rho}{\rho_s}\right)^2\geqslant 0, \tag{43}$$

which is both positive definite and of second order in the deviation from the steady state, thereby fulfilling the principal condition required on entropy production. On the other hand, the first term on the right-hand side of Eq. (42), $\overline{\mathrm{div}F^t}$, has no definite sign and contains, in principle, contributions of all orders in the deviation from steady state. In the stationary state, $dS/dt=0$, and the contribution of this term and of the second one in Eq. (42) must cancel that of $\Delta\dot{S}_P$. The role of this latter term in this balance is, then, to remove the contributions of all but second orders in the deviation from steady state contained in $\overline{\mathrm{div}F^t}$.

We may therefore write, in the new steady state

$$\Delta\dot{S}_P=-\overline{\mathrm{div}F^\infty}-(\text{terms of zeroth and first order in } h). \tag{44}$$

So by virtue of Eq. (29) we have

$$\Delta\dot{S}_P=-\sigma'-(\text{terms of zeroth and first order in } h). \tag{45}$$

This establishes a connection between the irreversible thermodynamics on the one hand, and phase-space dynamics on the other in the case when the dynamical system is externally driven by deterministic small term.

We now return to Eq. (6) and consider the dynamics in presence of an additional force hF_1 [Eq. (34)],

$$\frac{\partial\rho}{\partial t}=-\frac{\partial\phi\rho}{\partial U}-h\frac{\partial F_1\rho}{\partial U}+D_s\frac{\partial}{\partial U}\left(\rho_s\frac{\partial}{\partial U}\frac{\rho}{\rho_s}\right), \tag{46}$$

where ϕ is defined as

$$\phi=F_0-D_s\frac{\partial\ln\rho_s}{\partial U}. \tag{47}$$

Here we have used $\epsilon=1$ for the rest of the calculation.

The steady state solution of Eq. (6) is

$$\rho_s=N\exp\left[-\frac{\lambda U^2}{2D_s}\right], \tag{48}$$

where N is the normalization constant.

Using Eq. (48) in Eq. (47) we have

$$\phi\rho_s=0. \tag{49}$$

To consider the entropy flux and the entropy production in the nonequilibrium state in presence of external forcing we use Eq. (46) in the time evolution equation of entropy (11). Following Ref. [7] we finally identify entropy flux ($\Delta\dot{S}_F$) and entropy production ($\Delta\dot{S}_P$) as

$$\Delta\dot{S}_F=-\frac{d}{dt}\int\rho\frac{d\ln\rho_s}{dU}dU+\int\frac{dF_1}{dU}\delta\rho dU+\int dU\left(F_1\frac{d\ln\rho_s}{dU}\right)\delta\rho \tag{50}$$

and

$$\Delta\dot{S}_P=D_s\int dU\rho\left(\frac{d}{dU}\ln\frac{\rho}{\rho_s}\right)^2. \tag{51}$$

Here we have used $\delta\rho=\rho-\rho_s$ and $h=1$.

In the next step we solve Eq. (46) as before to find the explicit time dependence of $\Delta\dot{S}_F$ and $\Delta\dot{S}_P$. The time dependent solution of Eq. (46) is given by

$$\rho=N_1\exp\left[-\frac{[U-\beta_h(t)]^2}{\sigma(t)}\right], \tag{52}$$

where N_1 is the normalization constant and $\sigma(t)$ is obtained from Eq. (24). The expression for $\beta_h(t)$ is given by

$$\beta_h(t)=\frac{F_1}{\lambda}+\left(\beta_h(0)-\frac{F_1}{\lambda}\right)\exp[-\lambda t]. \tag{53}$$

Now using Eqs. (48) and (52) in both Eqs. (50) and (51) we have

$$\Delta\dot{S}_F=\frac{\lambda}{2D_s}[2D_s-\lambda\sigma(t)+2\beta_h(-\beta_h\lambda+F_1)]-\frac{\lambda}{D_s}F_1\beta_h \tag{54}$$

and

$$\Delta\dot{S}_P=D_s\left[\left(\frac{\lambda}{D_s}-\frac{2}{\sigma(t)}\right)\left\{\left(\frac{\lambda}{D_s}-\frac{2}{\sigma(t)}\right)\left(\beta_h^2+\frac{\sigma(t)}{2}\right)+4\frac{\beta_h^2}{\sigma(t)}\right\}+4\left(\frac{\beta_h}{\sigma}\right)^2\right], \tag{55}$$

where λ, D_s, $\sigma(t)$, $\beta_h(t)$, and F_1 are given by the Eqs. (10), (9), (24), (53), and (39), respectively. The time dependence of $\Delta\dot{S}_P$ is shown in Fig. 2 for different values of τ for a given set of values of other parameters. It is interesting to note that for $\gamma\tau\neq 1$ the entropy production first decreases with time and then passes through the minima and finally reaches to the following steady value [8] that is shown by solid curve of Fig. 2:

$$\Delta\dot{S}_P=\frac{(1-\gamma\tau)^2 f_e^2}{D}=-\Delta\dot{S}_F. \tag{56}$$

This observation can be explained by simplifying Eq. (55) in the limit $\sigma(0)\to 0$ and $\beta_h(t)\to 0$ as

$$\Delta\dot{S}_P=\frac{1}{D[1-\exp(-2\gamma t)]}[(1-\gamma\tau)^2 f_e^2(1-2e^{-\gamma t}+2e^{-3\gamma t}-e^{-4\gamma t})+\gamma De^{-4\gamma t}]. \tag{57}$$

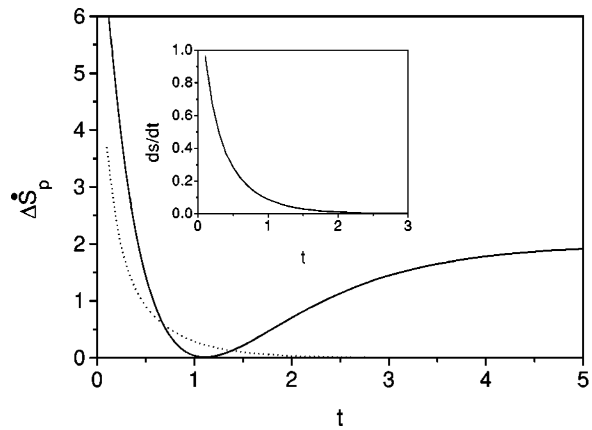

FIG. 2. Plot of entropy production $\Delta\dot{S}_P$ vs time using Eq. (55) for the same parameter set as in Fig. 1 and $\beta_h(0)=1.0$ and $f_e=1.0$. $\tau=2$, and 1 for solid and dotted curves. In the inset the sum of $\Delta\dot{S}_P$ and $\Delta\dot{S}_F$ from Eqs. (55) and (54) is plotted against time for $\tau=2$ (units are arbitrary).

In Eq. (57) first term in the numerator, which vanishes as $t\to 0$, implies that the external force increases entropy production while the second term corresponds to the decrease of entropy production with time due to dissipative action. Because of these two opposite effects a system thrown away from a steady state by a small external force relaxes to a new steady state passing through a minima in entropy production with time for the case $\gamma\tau\neq 1$. For $\gamma\tau=1$ entropy production decreases monotonically since the effective external force becomes zero under this condition. Similarly entropy flux also shows extremum properties for $\gamma\tau\neq 1$ case, which is shown in the solid curve of Fig. 3. Dotted curve of this figure corresponds to the time dependence of entropy flux for $\gamma\tau=1$. Another interesting point that should be noted here is that dS/dt and $\Delta\dot{S}_P$ or $\Delta\dot{S}_P$ reach their equilibrium values at different times (the plot of dS/dt vs t is shown in the inset of Fig. 2). Thus Fig. 2 implies that before the true stationary state is reached the system may show $dS/dt=0$.

In the Markovian limit $\tau\to 0$ so that Eq. (57) reduces to

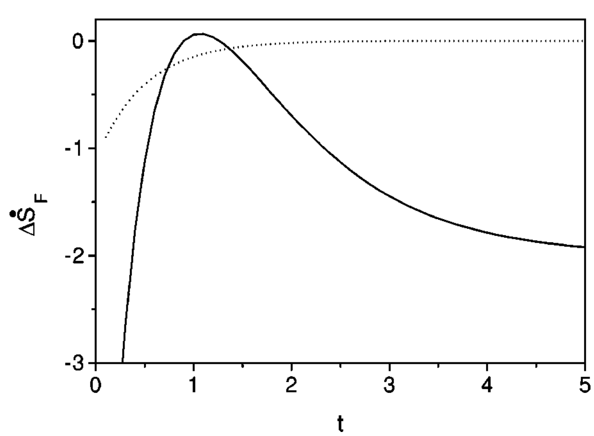

FIG. 3. Plot of entropy flux $\Delta\dot{S}_F$ vs time using Eq. (54) for the same parameter set as in Fig. 2. $\tau=2$ and 1 for solid and dotted curves (units are arbitrary).

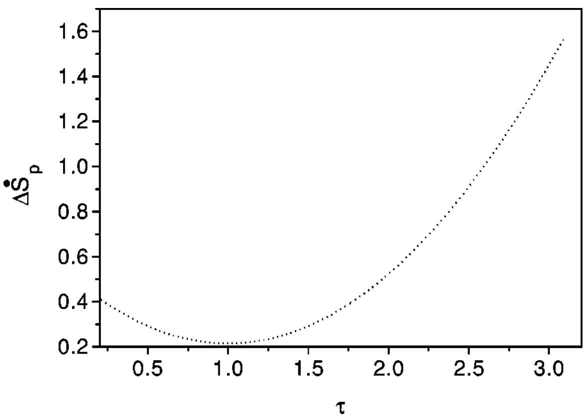

FIG. 4. Plot of entropy production ($\Delta\dot{S}_P$) vs τ using Eq. (55) for the same parameter set as in Fig. 2 at $t=0.5$ (units are arbitrary).

$$\Delta\dot{S}_P=\frac{1}{D[1-\exp(-2\gamma t)]}[f_e^2(1-2e^{-\gamma t}+2e^{-3\gamma t}-e^{-4\gamma t}) + \gamma D e^{-4\gamma t}]. \tag{58}$$

The above equation implies that even for white noise entropy production passes through the minima with time for both thermodynamically open and closed ($D=\gamma KT$) systems [17]. As $t\to\infty$ Eq. (58) reduces to

$$\Delta\dot{S}_P=\frac{f_e^2}{D}. \tag{59}$$

For $D=\gamma KT$ the above equation reduces to the standard result for entropy production of irreversible processes for a Brownian oscillator.

Equation (57) further implies that for $t>0$ the entropy production $\Delta\dot{S}_P$ passes through minimum at $\gamma\tau=1$, which is shown in Fig. 4. The variation of $\Delta\dot{S}_F$ with τ in Eq. (54) shows the maximum as evident in Fig. 5. These extremal behavior is not observed for $h=0$.

Now to show the effect of γ on the interplay between γ and τ we plot both $\Delta\dot{S}_P$ vs γ and $\Delta\dot{S}_F$ vs γ using Eqs. (55) and (54) (Figs. 6 and 7). Both the figures show extremum properties but Eqs. (26) and (27) do not exhibit such kind of variation. It is thus apparent that in presence of the nonequilibrium constraint the properties of noise processes as well as the dynamical characteristic of the system are important for both entropy flux and production.

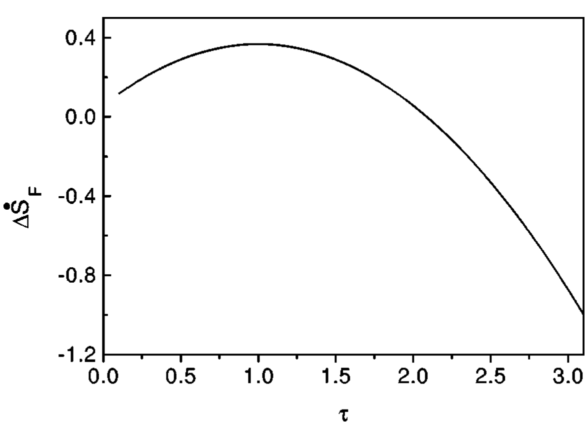

FIG. 5. Plot of entropy flux $\Delta\dot{S}_F$ vs τ using Eq. (54) for the same parameter set as in Fig. 2 at $t=0.5$ (units are arbitrary).

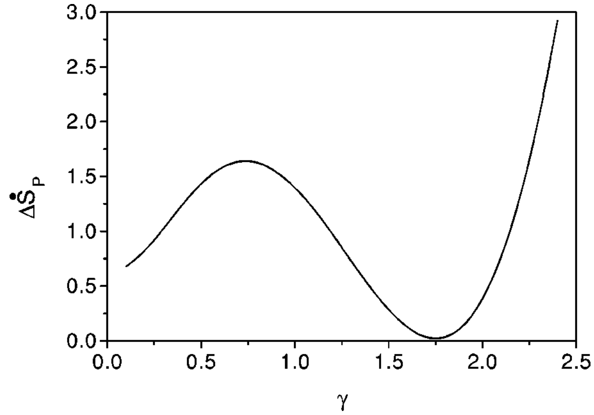

FIG. 6. Plot of entropy production $\Delta\dot{S}_P$ vs γ using Eq. (55) for the same parameter set as in Fig. 2 and $\tau=2$ at $t=0.5$ (units are arbitrary).

2. Cross-correlated noise-driven process

We now turn again to the cross-correlated noise-driven process to study the time dependence of entropy flux and entropy production due to additional weak forcing on the stationary system. To this end we add a constant of force f_e in Eq. (30),

$$\dot{X}=-\gamma X-\zeta_1 X+\eta_1+hf_e. \tag{60}$$

The Fokker-Planck equation corresponding to Eq. (60) can be written as

$$\frac{\partial\rho}{\partial t}=-\frac{\partial\phi_1\rho}{\partial X}-h\frac{\partial f_e\rho}{\partial X}+D_1\frac{\partial}{\partial X}\left(\rho_s\frac{\partial}{\partial X}\frac{\rho}{\rho_s}\right), \tag{61}$$

where

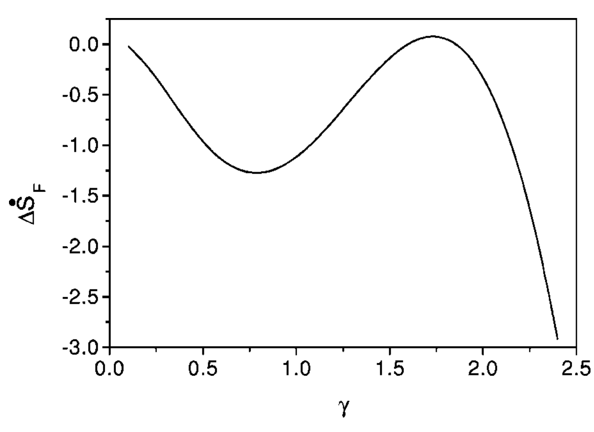

FIG. 7. Plot of entropy flux $\Delta\dot{S}_F$ vs τ using Eq. (54) for the same parameter set as in Fig. 6 at $t=0.5$ (units are arbitrary).

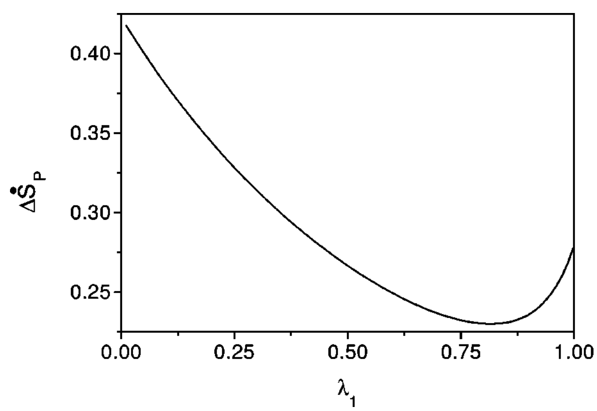

FIG. 8. Plot of entropy production $\Delta\dot{S}_P$ vs λ_1 using Eq. (65) for $\sigma_1=0.0$, $\beta_h'=0.0$, $D'=1.0$, $\alpha=1.0$, and $\gamma=1.0$ at $t=0.5$ (units are arbitrary).

$$\phi_1=F-D_1\frac{\partial \ln \rho_s}{\partial X}. \tag{62}$$

F is given by Eq. (33) and ρ_s is stationery solution of Eq. (32). Using ρ_s in $\phi_1\rho_s$ again we have

$$\phi_1\rho_s=0. \tag{63}$$

Since Eq. (61) is very much similar to the Eq. (32), the time dependence of entropy flux and entropy production can be derived as before to obtain

$$\Delta\dot{S}_F=\frac{\Gamma}{2D_1}\left[2D_1-\Gamma\sigma_1(t)+2\left(\beta_h'-\frac{l}{\Gamma}\right)(-\beta_h'\Gamma+l+f_e)\right]+\frac{lf_e}{D_1}-\frac{\Gamma}{D_1}f_e\beta_h' \tag{64}$$

and

$$\Delta\dot{S}_P=D_1\left[\left(\frac{\Gamma}{D_1}-\frac{2}{\sigma_1(t)}\right)\left\{\left(\frac{\Gamma}{D_1}-\frac{2}{\sigma_1(t)}\right)\left(\beta_h'^2+\frac{\sigma_1(t)}{2}\right)+2\left(\frac{2\beta_h'}{\sigma_1(t)}-\frac{l}{D_1}\right)\beta_h'\right\}+\left(2\frac{\beta_h'}{\sigma_1}-\frac{l}{D_1}\right)^2\right], \tag{65}$$

where

$$\beta_h'(t)=\left(\beta_h'(0)-\frac{l+f_e}{\Gamma}\right)\exp(-\Gamma t)+\frac{l+f_e}{\Gamma}. \tag{66}$$

Equations (65) and (64) also show extremal properties as shown by solid curves in Figs. 2 and 3. The variation of both $\Delta\dot{S}_P$ and $\Delta\dot{S}_F$ with noise correlation strength λ_1 in Eqs. (65) and (64) is shown in Figs. 8 and 9 respectively at $t=0.5$. Although both the figures show extremal behavior in the nonequilibrum state but at the stationary state $\Delta\dot{S}_P$ increases and $\Delta\dot{S}_F$ decreases monotonically. Thus the interplay between γ, noise strength and cross-correlation strength in the nonequilibrium state is different from that in the stationary state. Before leaving this section we mention here that our calculated entropy flux and entropy production are exact since the models considered here are linear and are exactly solvable by Greens' function of Gaussian form.

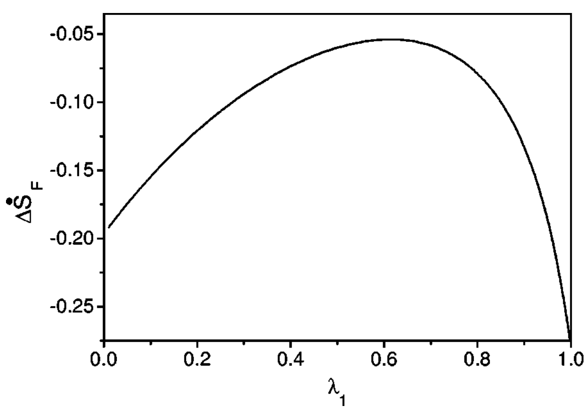

FIG. 9. Plot of entropy flux $\Delta\dot{S}_F$ vs λ_1 using Eq. (64) for the same parameter set as in Fig. 8 at $t=0.5$ (units are arbitrary).

III. CONCLUSIONS

In this paper we have explored the interplay between dissipative characteristics of the dynamics and noise properties in presence and absence of nonequilibrium constraint in the nonequilibrium state as well as in the stationary state in terms of entropy flux and entropy production. Both the entropy production and the entropy flux show extremal properties with time for color noise processes when the product of correlation time and dissipation constant is not equal to one in presence of a nonequilibrium constraint. The white and the cross-correlated noise-driven processes also mimic this extremal nature. This is due to a competition between the nonequilibrium constraint and the dissipative action. The maxima and minima are also found in the variation of both $\Delta\dot{S}_F$ and $\Delta\dot{S}_P$ with correlation time and dissipation constant for the color noise-driven processes in the nonstationary and the stationary states but this feature can be found in the variation of $\Delta\dot{S}_F$ and $\Delta\dot{S}_P$ as a function of correlation strength λ_1 only in the nonstationary state. Since white, color, or cross-correlated noise-driven processes concern many situations in biology, physics, and chemistry we hope that our present observation will be useful for understanding the close connection between irreversible thermodynamics and dynamical system in many related issues.

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[1] A.K. Pattanayak, Phys. Rev. Lett. **83**, 4526 (1999).
[2] J. Vollmer, T. Tél, and W. Breymann, Phys. Rev. E **58**, 1672 (1998).
[3] N.I. Chernov, G.L. Eyink, J.L. Lebowitz, and Ya.G. Sinai, Commun. Math. Phys. **154**, 569 (1993).
[4] A. Compte and D. Jou, J. Phys. A **29**, 4321 (1996).
[5] P. Gaspard, J. Stat. Phys. **88**, 1215 (1997).
[6] B.C. Bag, J. Ray Chaudhuri, and D.S. Ray, J. Phys. A **33**, 8331 (2000).
[7] D. Daems and G. Nicolis, Phys. Rev. E **59**, 4000 (1999).
[8] B.C. Bag, S. K. Banik, and D.S. Ray, Phys. Rev. E **64**, 026110 (2001).
[9] B.C. Bag, Phys. Rev. E **65**, 046118 (2002).
[10] B.C. Bag and D.S. Ray, J. Stat. Phys. **96**, 271 (1999).
[11] S. Chaudhuri, D. Majumdar, and D.S. Ray, Phys. Rev. E **53**, 5816 (1996).
[12] See, for example, H. Risken, *Fokker-Planck Equation* (Springer-Verlag, Berlin, 1989).
[13] J.H. Li and Z.Q. Huang, Phys. Rev. E **53**, 3315 (1996).
[14] J.H. Li and Z.Q. Huang, Phys. Rev. E **57**, 3917 (1998).
[15] G. Nicolis and D. Daems, J. Phys. Chem. **100**, 19 187 (1996).
[16] G. Nicolis and D. Daems, Chaos **8**, 311 (1998).
[17] K. Lindenberg and B. J. West, *The Nonequilibrium Statistical Mechanics of Open and Closed Systems* (VCH, New York, 1990).